

A bracket polynomial for graphs. III. Vertex weights

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Abstract

In earlier work the Kauffman bracket polynomial was extended to an invariant of marked graphs, i.e., looped graphs whose vertices have been partitioned into two classes (marked and not marked). The marked-graph bracket polynomial is readily modified to handle graphs with weighted vertices. We present formulas that simplify the computation of this weighted bracket for graphs that contain twin vertices or are constructed using graph composition, and we show that graph composition corresponds to the construction of a link diagram from tangles.

Keywords. graph, virtual link, Kauffman bracket, vertex weight, series, parallel, twin vertex, tangle, graph composition

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1 Introduction

In this paper a *graph* $G = (V(G), E(G))$ may have loops or multiple edges; it may also contain free loops, which are connected components that contain neither vertices nor edges. A *marked graph* is a graph whose vertex-set has been partitioned into two subsets, either of which may be empty. The vertices in one cell of the partition are *unmarked* and the vertices in the other cell are *marked*; in figures we indicate marked vertices with the letter c . If G is a marked graph with $V(G) = \{v_1, \dots, v_n\}$ then the *Boolean adjacency matrix* $\mathcal{A}(G)$ is an $n \times n$ matrix over the two-element field $GF(2)$, with entries $\mathcal{A}(G)_{ii} = 1$ if v_i is looped and for $i \neq j$, $\mathcal{A}(G)_{ij} = 1$ if v_i and v_j are adjacent. For $T \subseteq V(G)$ we denote by $\mathcal{A}(G)_T$ the matrix obtained from $\mathcal{A}(G)$ by first changing the i^{th} diagonal entry whenever $v_i \in T$, and then removing the i^{th} row and column whenever v_i is marked and the i^{th} diagonal entry is 0. The *marked-graph bracket polynomial* of G is defined by the formula

$$[G] = d^\phi \cdot \sum_{T \subseteq V(G)} A^{n-|T|} B^{|T|} d^{\nu(\mathcal{A}(G)_T)},$$



Figure 1: Signs and smoothings of classical crossings.

where ϕ is the number of free loops in G and ν is the $GF(2)$ -nullity of a matrix [37, 38].

An *oriented regular link diagram* D consists of oriented, piecewise smooth, closed curves in the plane; the curves intersect (and self-intersect) only at a finite number of transverse *crossings*. There are two kinds of crossings, classical and virtual, with underpassing and overpassing arcs specified at the classical crossings. A classical crossing has two *smoothings*, one denoted A and the other denoted B , as in Figure 1. If D has n classical crossings then it has 2^n *Kauffman states*, obtained by applying either the A or the B smoothing at each classical crossing. Given a state S let $a(S)$ denote the number of A smoothings in S , $b(S) = n - a(S)$ the number of B smoothings in S , and $c(S)$ the number of closed curves in S , including any crossing-free components that might appear in D . Then the (three-variable) *Kauffman bracket polynomial* of D [21, 23, 24] is

$$[D] = \sum_S A^{a(S)} B^{b(S)} d^{c(S)-1}.$$

The fact that A , B and d are independent variables is indicated by using $[D]$ rather than the more familiar notation $\langle D \rangle$.

These two kinds of bracket polynomials are closely related. A link diagram D has an associated *directed universe* \vec{U} , a 2-in, 2-out digraph whose vertices correspond to the classical crossings of D and whose edges correspond to the arcs of D . (\vec{U} also contains a free loop for each link component that is crossing-free in D .) The undirected version of \vec{U} is denoted U . Let C be a directed Euler system for \vec{U} , i.e., a set containing one directed Euler circuit for each connected component of U ; C must also contain every free loop of U . C is completely determined by specifying the classical crossings at which it does not follow the incident link component(s). We say such crossings are *marked*, and we indicate them in figures with the letter c . The *looped interlacement graph* $\mathcal{L}(D, C)$ [37, 38] is a marked graph with a vertex for each classical crossing of D ; marked vertices correspond to marked crossings, and looped vertices correspond to negative crossings. Two vertices v and w are adjacent in $\mathcal{L}(D, C)$ if and only if they are *interlaced* with respect to C , i.e., there is a circuit of C on which they appear in the order $v \dots w \dots v \dots w$ [29]. Different choices of C may give rise to different graphs $\mathcal{L}(D, C)$, as in Figure 2, but every looped interlacement graph has marked-graph bracket $[\mathcal{L}(D, C)]$ equal to the Kauffman bracket $[D]$.

The identity $[\mathcal{L}(D, C)] = [D]$ is derived from an equality connecting nulli-

ties of matrices over $GF(2)$ with circuits in 4-regular graphs: if S is a Kauffman state in a link diagram D and $T \subseteq V(\mathcal{L}(D, C))$ consists of those vertices where S involves the B smoothing, then the equality tells us that $c(S) - 1 = \phi + \nu(\mathcal{A}(\mathcal{L}(D, C))_T)$. Several similar equalities have been discovered independently over the years. This form of the equality originated as a result about permutations due to Cohn and Lempel [11], but related results appeared much earlier, in Brahana's study of curves on surfaces [6]. Later authors worked in combinatorics [3, 26] or classical knot theory [27, 31, 39]. See [35] for a thorough account.

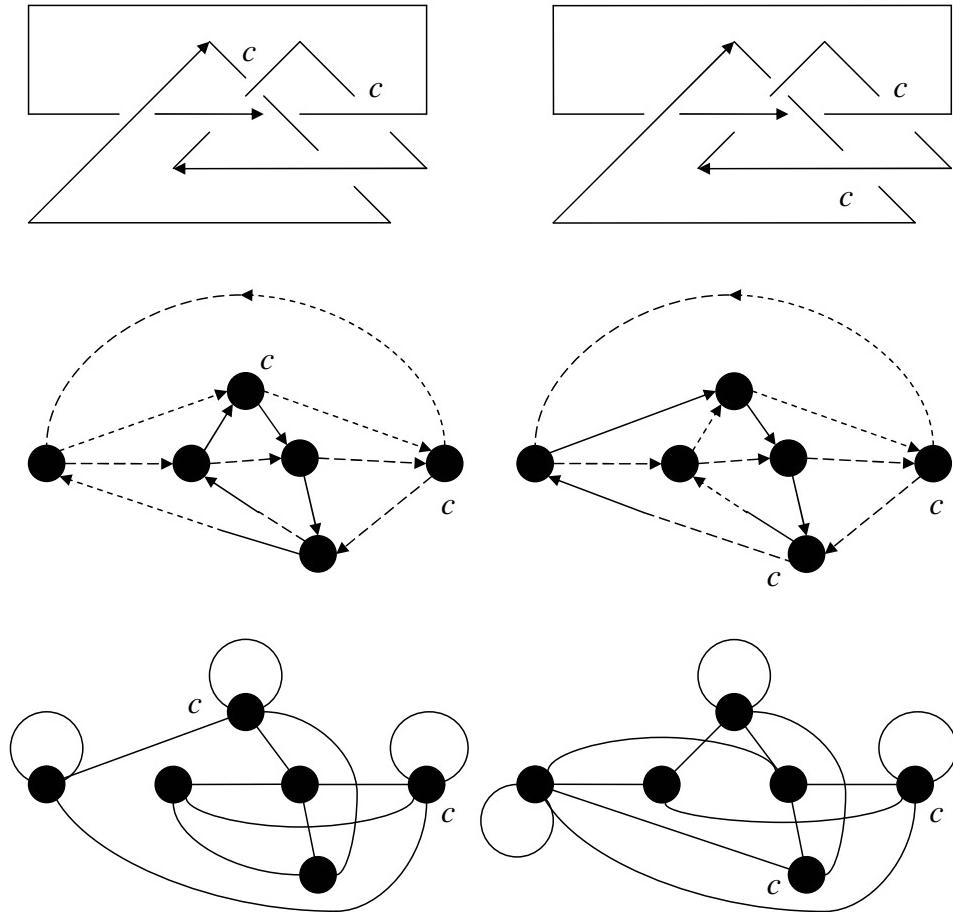


Figure 2: Two marked diagrams of the Borromean rings, the corresponding directed universes with Euler circuits indicated by dashes, and the associated looped interlace graphs.

For alternating classical link diagrams, the equality connecting nullity and

circuits appears implicitly in the relationship between the Kauffman bracket and certain combinatorial invariants: the Tutte polynomial [33], Jaeger's transition polynomial [16, 18], or the interlace polynomial [1]. The Tutte polynomial involves the checkerboard graph, and the latter two polynomials involve a directed graph obtained from the universe of a link diagram by reversing the orientation of every second edge. These relationships also hold for non-alternating classical diagrams, if weighted edges or vertices are used to distinguish between positive and negative classical crossings [22, 28, 32, 34, 36]. Since Kauffman introduced virtual knot theory [23], the combinatorial theory associated with the Kauffman bracket of classical link diagrams has been extended to virtual link diagrams in several ways. For checkerboard-colorable virtual links, little modification of the classical theory is needed [20]. For general virtual links there is not a direct analogue of the checkerboard graph or the edge-reversed universe graph, but the standard Tutte polynomial may be replaced by a "topological" Tutte polynomial [4, 5, 9, 10], or by a "relative" or "ported" Tutte polynomial [8, 15]. Kauffman's bracket may also be analyzed combinatorially without using any version of the Tutte polynomial; Ilyutko and Manturov have developed a geometric approach involving atoms and rotating circuits [17], and Zulli and the present author have developed a graph-theoretic approach using Euler circuits, interlacement and marked graphs [37, 38].

Marked graphs are subject to two important equivalence relations, which generalize natural equivalences between looped interlacement graphs. The finer of the two equivalence relations generalizes the equivalence between looped interlacement graphs $\mathcal{L}(D, C_1)$ and $\mathcal{L}(D, C_2)$ obtained from a single link diagram D . This equivalence relation is generated by a graph-theoretic operation we call a *marked pivot*; it is a modified version of the pivot operation that describes the effect on interlacement graphs of changing Euler systems in 2-in, 2-out digraphs [1, 25]. Marked pivots preserve the 3-variable marked-graph bracket polynomial. The coarser of the two equivalence relations generalizes the equivalence between looped interlacement graphs $\mathcal{L}(D_1, C_1)$ and $\mathcal{L}(D_2, C_2)$ obtained from different diagrams of the same link type. This equivalence relation is generated by graph-theoretic versions of the Reidemeister moves. Marked-graph Reidemeister moves change the 3-variable marked-graph bracket, but they preserve the marked-graph analogue of the Jones polynomial, which is obtained (as usual) by replacing A with $t^{-1/4}$, replacing B with $t^{1/4}$, replacing d with $-t^{1/2} - t^{-1/2}$, and multiplying by a suitable factor.

These two equivalence relations are certainly important, for they underlie the knot-theoretic significance of the marked-graph bracket. Nevertheless we pay little attention to them in this paper, because our purpose is not to discuss the relationship between bracket polynomials of different graphs, but rather to discuss the efficient computation of the bracket polynomial of a given graph.

The formulas that define the marked-graph bracket and the three-variable Kauffman bracket are quite similar, so it may be surprising that recursive descriptions of the two bracket polynomials are quite different. The Kauffman bracket of a link diagram D can be calculated by repeatedly applying a single recursive step: choose a classical crossing in D , let D_A and D_B be

the two diagrams obtained by smoothing that crossing, and use the formula $[D] = A[D_A] + B[D_B]$. The recursive description of the marked-graph bracket given in [37, 38] is considerably more complicated; four different recursive steps are used, in different situations. For instance, one step removes a loop on an unmarked vertex, at the cost of replacing the graph in question with two smaller graphs; this particular step is related to the Jones polynomial's fundamental identity $tV_{L^-} = t^{-1}V_{L^+} - (t^{1/2} - t^{-1/2})V_L$ [19].

If we rewrite the definition of the marked-graph bracket polynomial as

$$[G] = d^\phi \cdot \sum_{T \subseteq V(G)} (\prod_{v \notin T} A)(\prod_{t \in T} B) d^{\nu(\mathcal{A}(G)_T)},$$

then the variables A and B appear as vertex weights, similar to those that appear in many combinatorial contexts ranging from electrical circuit theory to statistical mechanics. (For instance, if $A = B = \frac{1}{2}$ then $[G]$ gives the expected value of $d^{\phi+\nu(\mathcal{A}(G)_T)}$ under the presumption that T is chosen by tossing a fair coin n times, the i^{th} toss deciding whether or not $v_i \in T$.) The following generalization suggests itself.

Definition 1 Suppose G is a weighted, marked graph, i.e., a marked graph given with functions α and β mapping $V(G)$ into some commutative ring R . Then the weighted marked-graph bracket polynomial of G is

$$[G] = d^\phi \cdot \sum_{T \subseteq V(G)} (\prod_{v \notin T} \alpha(v))(\prod_{t \in T} \beta(t)) d^{\nu(\mathcal{A}(G)_T)}$$

If $v \in V(G)$ has $\alpha(v) = A$ and $\beta(v) = B$ then we say v has *standard weights*. If D is a link diagram then the vertices of $G = \mathcal{L}(D, C)$ correspond to the classical crossings of D , so we may think of α and β as giving weights for the classical crossings of D .

The recursive description of the marked-graph bracket polynomial given in [37] extends directly to the weighted version of the polynomial. Vertex weights may be used to make the recursion more efficient in several ways. The most obvious simplification involves the recursive step mentioned above, used to eliminate loops on unmarked vertices; it is completely unnecessary.

Theorem 2 Suppose G is a weighed, marked graph. Let G' be the graph obtained by removing every loop from G , and reversing the α and β weights of every looped vertex of G . Then $[G] = [G']$.

The value of Theorem 2 is easy to see: each time we use the theorem instead of a loop-removing recursive step, the recursion proceeds with only one graph to process rather than two.

The weighted marked-graph bracket polynomial also satisfies several analogues of the series-parallel reductions of electrical circuit theory. These are operations which consolidate certain vertices without changing the value of the bracket. Here is one of them.

Theorem 3 Suppose v_1, \dots, v_k are unlooped twins that form a clique in G , i.e., v_1, \dots, v_k all have the same neighbors outside $\{v_1, \dots, v_k\}$ and they are all adjacent to each other. Let $\rho = |\{i | v_i \text{ is not marked}\}|$. Let $(G - v_2 - \dots - v_k)'$ be the graph obtained from $G - v_2 - \dots - v_k$ by (i) marking v_1 if and only if ρ is even, and (ii) changing the weights of v_1 to

$$\alpha'(v_1) = \prod_{i=1}^k \alpha(v_i) \quad \text{and} \quad \beta'(v_1) = d^{-1} \left(-\alpha'(v_1) + \prod_{i=1}^k (\alpha(v_i) + d\beta(v_i)) \right).$$

Then $[G] = [(G - v_2 - \dots - v_k)']$.

The value of Theorem 3 is not quite as obvious as that of Theorem 2. A first impression might be that we are simply replacing k vertices with one vertex, but this impression is imprecise because the complicated values of $\alpha'(v_1)$ and $\beta'(v_1)$ given in the theorem may be inconvenient. The computational cost of this inconvenience depends on the implementation of arithmetic operations in the particular ring being used for R . For instance, a natural example is a ring R of polynomials in variables $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$, with the variables used as vertex-weights; arithmetic in this ring is very expensive because each polynomial involves coefficients of many different monomials. Nevertheless, Theorem 3 clearly has the potential to be of significant value in general.

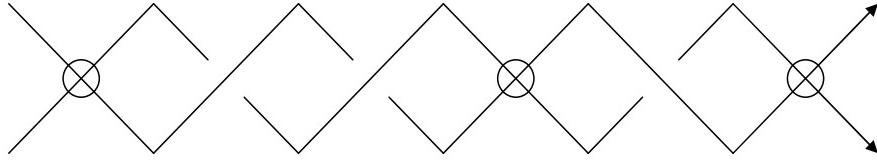


Figure 3: Twisted strands give rise to a clique of twin vertices in $\mathcal{L}(D, C)$.

The most familiar situation in which Theorem 3 arises involves two coherently oriented strands of a link diagram, twisted around each other to produce k classical crossings and some number of virtual crossings. Although the theorem specifies that the vertices are unlooped, negative crossings may be handled simply by reversing their α and β weights. An example with $k = 3$ appears in Figure 3. (As usual, the circled crossings are virtual.) If these twisted strands appear in a link diagram D with standard weights then the two negative crossings give rise to unlooped vertices of $\mathcal{L}(D, C)$ with $\alpha = B$ and $\beta = A$, while the positive crossing gives rise to an unlooped vertex with $\alpha = A$ and $\beta = B$. Theorem 3 tells us that the bracket is unchanged if the three vertices of $\mathcal{L}(D, C)$ representing the portion of D appearing in Figure 3 are replaced with one unlooped, unmarked vertex whose weights are $\alpha = AB^2$ and $\beta = (-AB^2 + (A+Bd)(B+Ad)^2)/d$. This new graph is the looped interlacement graph of a diagram D' obtained by replacing the pictured portion of D with a single positive crossing carrying the indicated weights, and also a single virtual crossing; the latter is needed

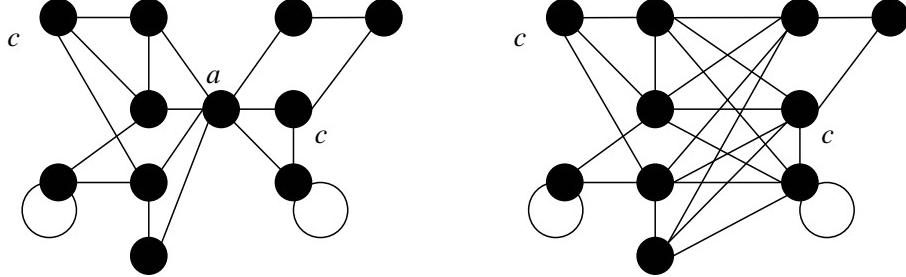


Figure 4: Composition of graphs.

whenever the total number of unmarked classical and virtual crossings is even, to ensure that C gives rise to an Euler system of D' .

In the classical case – or more generally, the checkerboard-colorable case [20] – twisting two strands around each other produces classical crossings that give rise to series-parallel edges in the checkerboard graphs. Theorem 3 applies to all virtual link diagrams, not just the checkerboard-colorable ones, but the extra generality comes at a price: the theorem must be adjusted when there are marks on the vertices or non-adjacencies among them, and not all of the adjusted versions are quite so simple. For instance, if two oppositely oriented strands of a link diagram are twisted around each other to produce a set of k unmarked, nonadjacent twins then a “dual” of Theorem 3 requires that k be odd. See Corollary 16.

A third use for vertex weights is that graphs constructed from smaller graphs using an appropriate version of Cunningham’s *composition* operation [13] have bracket polynomials that can be described by modifications of the weights.

Definition 4 A marked, weighted graph G is the composition of marked, weighted graphs F and H , $G = F * H$, if the following conditions hold.

- (a) $V(F) \cap V(H)$ consists of a single unlooped, unmarked vertex a that has standard weights in both F and H .
- (b) The elements of $V(G) = V(F) \cup V(H) - \{a\}$ inherit their loops, marks and weights from F and H .
- (c) $E(G) = E(F-a) \cup E(H-a) \cup \{\{v, w\} | \{v, a\} \in E(F) \text{ and } \{a, w\} \in E(H)\}$.
- (d) F and H do not share any free loop, and the free loops of G are those of F and H .

Requiring that a have standard weights and be unlooped and unmarked ensures that no significant information is lost when we remove a in constructing $F * H$. Note that Definition 4 includes the situation of Theorem 3: if F is a complete graph then $F - a$ is a clique of twins in $F * H$.

The construction given in Definition 4 may seem to be merely a technical notion from graph theory, but in Section 2 we show that it is related to an

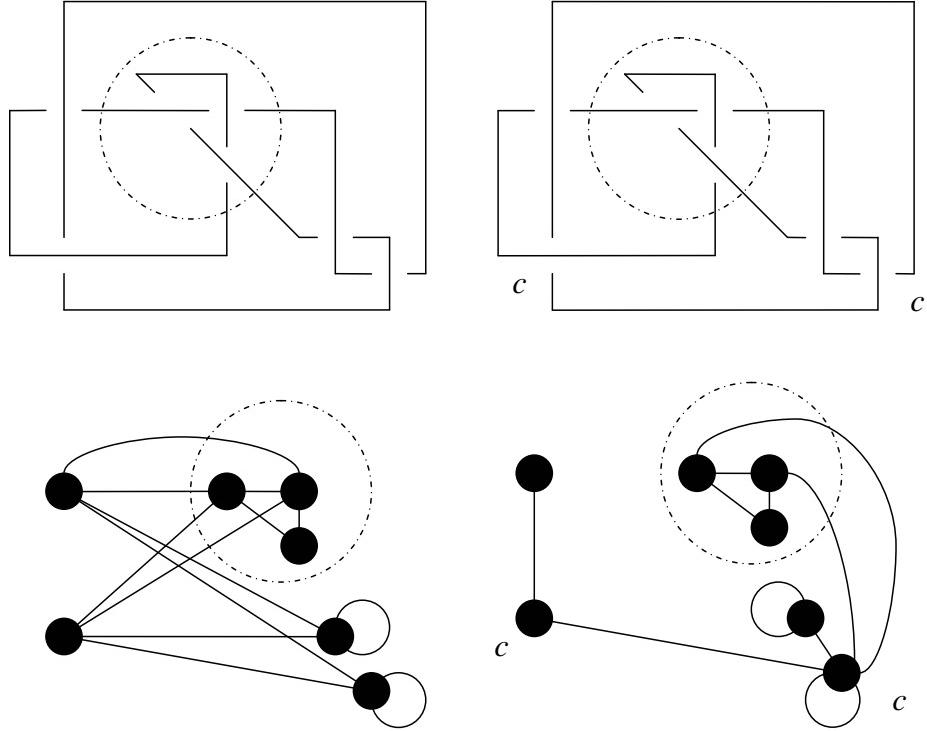


Figure 5: If D contains a tangle then $\mathcal{L}(D, C)$ is a composition of graphs.

important knot-theoretic idea: if a link diagram contains a tangle, then the looped interlacement graph is a composition. Recall that a subgraph of a graph G is *full* or *induced* if it contains every edge of G incident on its vertices.

Theorem 5 Suppose a link diagram D contains a tangle, and let C be any directed Euler system for the universe of D . Then there are graphs F and H such that $\mathcal{L}(D, C) = F * H$, $F - a$ is the subgraph of $\mathcal{L}(D, C)$ induced by the vertices corresponding to crossings outside the tangle, and $H - a$ is the subgraph of $\mathcal{L}(D, C)$ induced by the vertices corresponding to crossings inside the tangle. Moreover, it is possible to choose C so that at least one of F, H has no marked vertex adjacent to a .

Two instances of Theorem 5 are pictured in Figure 5. Observe that D contains several tangles in addition to the one indicated by the dashed circle. The interlacement graph on the left satisfies the last sentence of the theorem for every tangle, as no vertex is marked. The interlacement graph on the right, instead, satisfies the last sentence for some tangles but not for others. For instance, it satisfies the last sentence of the theorem for the indicated tangle,

but not for the tangle that contains the two crossings in the lower right-hand corner of the diagram.

For a fixed graph F , every composition $F * H$ is constructed from H in much the same way. In Section 5 we prove that similarly, every weighted bracket polynomial $[F * H]$ is constructed from bracket polynomials associated with H in much the same way.

Theorem 6 *Let F be a marked, weighted graph with an unlooped, unmarked vertex a that has standard weights. Then there are weights $\alpha'(a)$, $\beta'(a)$, and $\alpha'(a_m)$ that depend only on F and a , and have the following “universal” property: every composition $F * H$ obtained by applying Definition 4 to a graph H in which no neighbor of a is marked has*

$$[F * H] = [H'] + [H'_m],$$

where H' is obtained from H by changing the weights of a to $\alpha'(a)$ and $\beta'(a)$, and H'_m is obtained from H by marking a and changing its weights to $\alpha'(a_m)$ and $\beta'(a_m) = 0$.

Choosing to have $\beta'(a_m) = 0$ in Theorem 6 is a matter of convenience rather than necessity. The proof actually shows that the value of $\beta'(a_m)$ is arbitrary, but the sum $\beta'(a) + \beta'(a_m)$ must be correct. That is, for any $r \in R$ the theorem still holds if we change $\beta'(a_m)$ from 0 to r , and also change $\beta'(a)$ to $\beta'(a) - r$. In particular, Theorem 6 remains valid if $\beta'(a)$ and $\beta'(a_m)$ are interchanged.

It is hard to assess the loss of generality associated with Theorem 6’s hypothesis that a have no marked neighbor in H , because vertex marks do not have a special significance in general. For tangles in link diagrams, though, the last sentence of Theorem 5 tells us that this hypothesis does not entail any loss of generality as long as we are willing to reverse the roles of F and H , i.e., to reverse the “inside” and “outside” of the tangle.

The idea that tangles are building blocks for classical link diagrams appears in Conway’s seminal paper [12]. This idea leads naturally to the observation that a recursively defined classical link invariant can be calculated in a “tangle-based” manner: first eliminate all the crossings inside a particular tangle and then collect like terms, before proceeding to eliminate the crossings inside another tangle. See [30] for a detailed analysis of the computational complexity of such a tangle-based calculation of classical link invariants. The process of building up link diagrams from tangles corresponds to the process of building up checkerboard graphs as 2-sums of smaller graphs, and this correspondence is useful for the Kauffman bracket because the Tutte polynomial of a 2-sum can be described using the Tutte polynomials of the smaller graphs [7]. The correspondence between tangles and 2-sums was mentioned in [32], and its computational significance has recently been analyzed in [14]. The result of a tangle-based calculation of the Kauffman bracket of a classical diagram containing a tangle T may be represented schematically as $[T] = \gamma_1[\asymp] + \gamma_2[\sqcap]$, where γ_1 and γ_2 are coefficients that result from the collection of terms. Virtual crossings necessitate a third term; schematically, $[T] = \gamma_1[\asymp] + \gamma_2[\sqcap] + \gamma_3[\otimes]$. At first glance this

three-term schematic formula may seem different from the two-term formula of Theorem 6, but they are actually quite similar; each has three degrees of freedom, represented by $\gamma_1, \gamma_2, \gamma_3$ in the schematic formula and $\alpha(a), \beta(a), \alpha(a_m)$ in Theorem 6.

Formulas for $\alpha(a)$, $\beta(a)$, and $\alpha(a_m)$ are presented in Corollaries 7, 11 and 21. The first two are obtained by using several specific graphs for H , and then solving the resulting equations. The third breaks Definition 1 into three “sub-sums.”

Suppose F is a marked, weighted graph with an unlooped, unmarked vertex a that has standard weights. For $i \neq j \in \{0, 1\}$ let F^{ij} be the graph obtained from F by replacing a with a vertex v_{ij} whose weights are $\alpha(v_{ij}) = i$ and $\beta(v_{ij}) = j$. Then $F - a$, F^{10} and F^{01} are compositions $F * H^0$, $F * H^{10}$ and $F * H^{01}$ respectively, where H^0 is just a and H^{ij} has the two adjacent, unlooped, unmarked vertices a and v_{ij} . Theorem 6 gives three equations.

$$\begin{aligned} [F - a] &= [F * H^0] = [H^{0'}] + [H_m^{0'}] = d\alpha(a) + \beta(a) + \alpha(a_m) \\ [F^{10}] &= [F * H^{10}] = [H^{10'}] + [H_m^{10'}] = \alpha(a) + \beta(a) + d\alpha(a_m) \\ [F^{01}] &= [F * H^{01}] = [H^{01'}] + [H_m^{01'}] = \alpha(a) + d\beta(a) + \alpha(a_m) \end{aligned}$$

We deduce the following.

Corollary 7 *The weights mentioned in Theorem 6 are given by these formulas.*

$$\begin{aligned} (2 - d - d^2)\alpha(a) &= -(d + 1)[F - a] + [F^{10}] + [F^{01}] \\ (2 - d - d^2)\beta(a) &= [F - a] + [F^{10}] - (d + 1)[F^{01}] \\ (2 - d - d^2)\alpha(a_m) &= [F - a] - (d + 1)[F^{10}] + [F^{01}] \end{aligned}$$

To assess the computational significance of Theorem 6 and Corollary 7, consider that the number of steps in an implementation of the recursive algorithm for calculating $[G]$ is roughly $2^{|V(G)|}$. (This is only a rough count rather than a precise determination of computational complexity, because it ignores both the computational cost of arithmetic in R and the fact that different branches of a calculation may require different numbers of steps.) Consequently the number of steps involved in a direct computation of $[F * H]$ is roughly the product of the numbers of steps involved in separate computations of $[F - a]$ and $[H - a]$. Corollary 7 shows that the weights of a and a_m may be calculated using the bracket polynomials of $F - a$ and two graphs that have $|V(F)|$ vertices apiece, so we may estimate the number of steps involved in finding these weights as roughly five times the number of steps involved in finding $[F - a]$. Theorem 6 then expresses $[F * H]$ as the sum of the bracket polynomials of two $|V(H)|$ -vertex graphs; computing each of these brackets takes roughly twice as many steps as computing $[H - a]$. All in all, Theorem 6 and Corollary 7 tell us that a rough upper bound on the number of steps required to compute $[F * H]$ is on the order of five times the sum of the numbers of steps required to calculate $[F - a]$ and $[H - a]$ separately. Five times the sum of two positive integers is

generally considerably smaller than the product of the two integers, so despite the rough counting it is clear that when they are applicable, Theorem 6 and Corollary 7 can be considerably more efficient than direct computation.

Theorem 6 focuses on F , but compositions are symmetric and consequently the theorem may be applied to both F and H , so long as neither contains a marked neighbor of a . Unlike the hypothesis of Theorem 6 that only H have no marked vertex adjacent to a , this double hypothesis is a significant restriction even for link diagrams. The result is still useful, though; for instance, every classical or virtual knot diagram has an Euler system with respect to which there are no marked vertices at all.

Corollary 8 *Let $a(F)$ and $a_m(F)$ be the new weighted vertices associated to F and a in Theorem 6, and let $a(H)$ and $a_m(H)$ be the new vertices associated in the same way to H and a . (That is, they are obtained by interchanging F and H in Theorem 6.) If neither F nor H contains a marked neighbor of a then*

$$\begin{aligned} [F * H] = & \alpha(a(F)) \cdot (\alpha(a(H)) + \beta(a(H))) + \beta(a(F)) \cdot (\alpha(a(H)) + \beta(a(H))d) \\ & + (\alpha(a(F))d + \beta(a(F))) \cdot \alpha(a_m(H)) \\ & + \alpha(a_m(F)) \cdot (\alpha(a(H))d + \beta(a(H))) + \alpha(a_m(F)) \cdot \alpha(a_m(H)). \end{aligned}$$

The corollary is proven as follows. Let P_1 be the graph with two adjacent vertices a and b , with b an unlooped, unmarked vertex carrying the weights of $a(F)$. Let P_{1m} have two adjacent vertices a and b_m with b_m an unlooped, marked vertex carrying the weights of $a_m(F)$. Three applications of Theorem 6 tell us that

$$\begin{aligned} [F * H] &= [H'] + [H'_m] = [H * P_1] + [H * P_{1m}] \\ &= [P'_1] + [(P'_1)_m] + [(P_{1m})'] + [(P_{1m})'_m]. \end{aligned}$$

Before proceeding we should express our gratitude to V. O. Manturov for his comments and corrections regarding earlier versions of the paper.

2 Diagrams and tangles

Suppose D is a diagram of a μ -component link $L = K_1 \cup \dots \cup K_\mu$, and let Γ denote the graph with $V(\Gamma) = \{v_1, \dots, v_\mu\}$ and $E(\Gamma) = \{\{v_i, v_j\}\}$ such that D contains a classical crossing involving K_i and K_j . Suppose C is a directed Euler system of \vec{U} . As we traverse one of the circuits of C , we pass from one link component to another when we encounter a marked vertex representing a classical crossing involving those two link components. Each circuit of C must cover all the link components that appear together in a connected component of Γ , so if Γ has $c(\Gamma)$ connected components then $\mathcal{L}(D, C)$ must have at least $\mu - c(\Gamma)$ marked vertices. It is a simple matter to construct an Euler circuit with precisely $\mu - c(\Gamma)$ marked vertices: choose a spanning forest in Γ , and mark one crossing of D corresponding to each edge of the spanning forest.

Lemma 9 *If $\mathcal{L}(D, C)$ has $\mu - c(\Gamma)$ marked vertices then no two of them are adjacent to each other.*

Proof. If two marked vertices v and w are neighbors then they are interlaced on a circuit of C . If the circuit is $C_1 = C_{11}vC_{12}wC_{13}vC_{14}w$ then $C_1 * v * w * v = C_{11}vC_{14}wC_{13}vC_{12}$ is also an Euler circuit of the same connected component of U . If we replace C_1 with $C_1 * v * w * v$ then we obtain an Euler system $C * v * w * v$ which has precisely the same marked vertices as C , except that v and w are no longer marked. This is impossible as $\mu - c(\Gamma)$ is the minimum number of marked vertices. ■

We are now ready to prove Theorem 5.

Suppose D contains a tangle (more precisely, a *2-tangle*), i.e., a portion of D that can be enclosed by a circle which intersects D in precisely four points, none of which is a crossing. Let p_1, p_2, p_3 and p_4 be the four intersection points of D and the tangle's boundary circle. Let C be a directed Euler system for \vec{U} , and consider a circuit $C_1 \in C$ that passes through p_1 ; suppose C_1 is directed into the circle at p_1 . Clearly C_1 must leave the circle, say at p_2 . There are two possibilities: either C_1 also passes through p_3 and p_4 , or a different circuit of C passes through them.

In the first case, suppose C_1 is $C_{11}p_1C_{12}p_2C_{13}p_3C_{14}p_4$, with C_{11} and C_{13} outside the circle. A vertex of $\mathcal{L}(D, C)$ corresponds to a classical crossing of D ; either the crossing is inside the tangle, in which case the vertex cannot appear on C_{11} or C_{13} , or else the crossing is outside the tangle, in which case the vertex cannot appear on C_{12} or C_{14} . If the crossing appears on more than one arc C_{1j} , then, it must appear either on C_{11} and C_{13} or on C_{12} and C_{14} . Clearly these two types of vertices are interlaced with respect to C_1 . Also, a vertex that appears only on a single C_{1j} cannot be interlaced with a vertex that does not appear on the same C_{1j} . Consequently the first assertion of Theorem 5 holds, with the vertices that appear on both C_{12} and C_{14} adjacent to a in H and the vertices that appear on both C_{11} and C_{13} adjacent to a in F .

In the second case, no vertex corresponding to a crossing inside the tangle is interlaced with any vertex corresponding to a crossing outside the tangle. Consequently $\mathcal{L}(D, C)$ is the disjoint union of $F - a$ and $H - a$, so it is a composition with a an isolated vertex.

This completes the proof of the first assertion of Theorem 5. To prove the second assertion, note that the lemma tells us that it is possible to choose C so that no two marked vertices of $\mathcal{L}(D, C)$ are neighbors. As every neighbor of a in F is adjacent in $F * H$ to every neighbor of a in H , at least one of F, H must contain no marked neighbor of a .

Corollary 10 *If D contains a tangle then Theorem 6 may be used to describe the Kauffman bracket of D using appropriately weighted versions of the subgraphs of $\mathcal{L}(D, C)$ induced by the vertices inside and outside the tangle.*

Proof. As discussed above, it is always possible to choose C so that no two marked vertices of $\mathcal{L}(D, C)$ are adjacent. Then Theorem 5 tells us that Theorem 6 applies, possibly with the names of F and H interchanged. ■

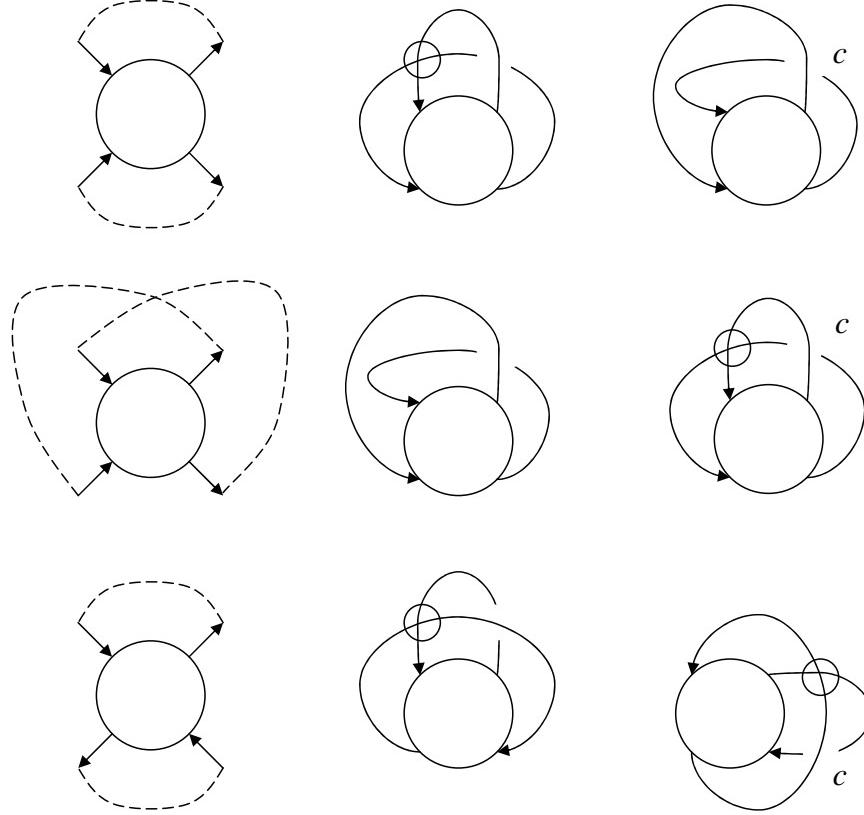


Figure 6: At left is a directed tangle in D , with dashes indicating the Euler system C . $H - a$ is the subgraph of $\mathcal{L}(D, C)$ induced by vertices corresponding to crossings inside the tangle. The center diagram corresponds to H' and the right-hand diagram corresponds to H'_m .

In the situation of Corollary 10, Theorem 6 tells us that $[D]$ is equal to the sum of the bracket polynomials of two graphs with weighted vertices. It turns out that these two graphs are looped interlacement graphs, so $[D]$ is actually equal to the sum of the bracket polynomials of two link diagrams with weighted crossings. Suppose D contains a directed tangle as illustrated on the left-hand side of Figure 6, and an Euler system C has been chosen so that the F, H notations of Theorems 5 and 6 agree. (The dashed arcs in the figure indicate the paths C might follow as it leaves and re-enters the tangle.) Then $F - a$ is the subgraph of $\mathcal{L}(D, C)$ induced by the vertices corresponding to the crossings outside the circle, $H - a$ is the subgraph of $\mathcal{L}(D, C)$ induced by the vertices corresponding to the crossings inside the circle, and the neighbors of a are the vertices of $F - a$ and $H - a$ that correspond to crossings outside and inside the

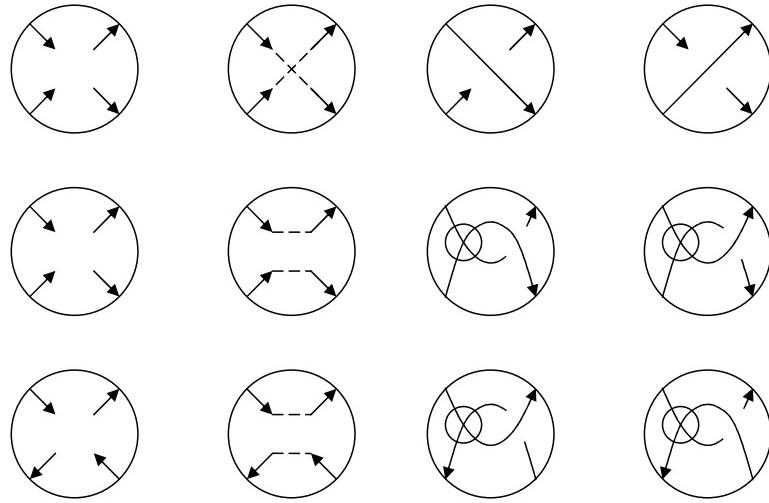


Figure 7: Each row represents a directed tangle located outside the circle, a dashed Euler system in D , and associated diagrams D_+ and D_- .

circle that are interlaced with each other. No neighbor of a in H is marked. The looped interlacement graphs of the diagrams shown in the center and on the right-hand side of each row of the figure are then isomorphic to the graphs H' and H'_m , with the new classical crossings corresponding to a and a_m respectively. (The figure does not completely specify these two weighted graphs, because the weights of a and a_m are not displayed.)

Observe that the portion of D inside the circle is not disturbed in the two new diagrams. Consequently if D is obtained by substituting tangles for the vertices of a 4-regular graph P (i.e., $D = P * t_1 \dots t_\tau$ in the notation of [12]), with t_τ containing the crossings outside the circle, then $t_1, \dots, t_{\tau-1}$ still appear as tangles in the two new diagrams, and Theorem 6 may be applied to $t_{\tau-1}, \dots, t_1$ in turn. The result of applying Theorem 6 repeatedly is to express $[D]$ as the sum of the bracket polynomials of 2^τ crossing-weighted τ -crossing diagrams. Such a sum seems complicated but depending on the structure of D , it may actually be considerably simpler than the definition of $[D]$.

We close this section with the observation that if F arises from a tangle in a link diagram then it is possible to replace the graphs F^{10} and F^{01} of Corollary 7 with graphs that also arise from link diagrams. Suppose we are given a directed tangle as on the left-hand side of Figure 7. Note that the convention of the figure is the opposite of the usual one – the crossings of the tangle correspond to the vertices of the subgraph $F - a$, so they are presumed to lie outside the circle; the four segments inside the circle are the ends of arcs of the tangle. The dashes in the second picture of each row indicate the path followed by C , and the third and fourth pictures in that row indicate virtual diagrams we

denote D_+ and D_- respectively. Let $F_+ = \mathcal{L}(D_+, C_+)$ and $F_- = \mathcal{L}(D_-, C_-)$, where C_+ and C_- are the Euler systems obtained from C in the obvious ways. Also, let $v_+ \in V(F_+)$ and $v_- \in V(F_-)$ be the vertices corresponding to the new classical crossings; then v_- is looped and v_+ is unlooped. If we weight v_+ and v_- with $\alpha(v_+) = \alpha(v_-) = A$ and $\beta(v_+) = \beta(v_-) = B$, it follows that $[F_+] = A[F^{10}] + B[F^{01}]$ and $[F_-] = B[F^{10}] + A[F^{01}]$. The formulas of Corollary 7 then imply the following.

Corollary 11

$$(2 - d - d^2)\alpha(a) = -(d + 1)[F] + \frac{[F_+] + [F_-]}{A + B}$$

$$(2 - d - d^2)\beta(a) = [F] + \frac{(A + B + Bd)[F_+] - (A + Ad + B)[F_-]}{A^2 - B^2}$$

$$(2 - d - d^2)\alpha(a_m) = [F] + \frac{(A + B + Bd)[F_-] - (A + Ad + B)[F_+]}{A^2 - B^2}$$

3 Twin vertices

Two edges of a graph incident on the same vertices are *parallel*, and two edges incident on a degree-two vertex are *in series*. Series and parallel edges in edge-weighted graphs can often be consolidated using some appropriate combination of weights; for instance, in electrical circuit theory two parallel resistors are equivalent to a single resistor with $R^{-1} = R_1^{-1} + R_2^{-1}$. Similarly, vertices v and w are called *twins* if they have the same neighbors outside $\{v, w\}$.



Figure 8: Crossings that are consecutive on both strands give rise to twin vertices in $\mathcal{L}(D, C)$.

Some twin vertices in interlacement graphs arise from simple configurations in link diagrams, like those pictured in Figure 8. Other twin vertices arise from more complicated configurations. For instance, let D_1 and D_2 be two link diagrams with connected universes, each marked to identify an Euler circuit. Suppose D_1 and D_2 are drawn together in the plane, with some finite, positive number of transverse intersections. Let D_3 be a link diagram obtained by considering each of these transverse intersections to be a classical crossing. Then

the universe U_3 of D_3 is connected, but the marks on D_1 and D_2 do not describe an Euler circuit in U_3 . If we locate a portion of D_3 where an arc of D_1 lies parallel to an arc of D_2 , and replace that portion of D_3 with the configuration pictured on the right in Figure 9, then the result is a marked link diagram D whose marks do describe an Euler circuit C . The two new classical crossings correspond to marked, nonadjacent twins in $\mathcal{L}(D, C)$.

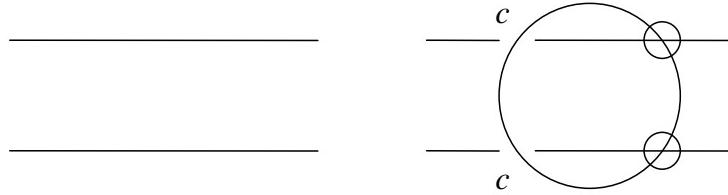


Figure 9: A construction that produces nonadjacent marked twins.

In some situations, the weighted bracket polynomial allows for the consolidation of twin vertices into one vertex.

Proposition 12 *Let $v, w \in V(G)$ be unlooped twins.*

(a) *Suppose v and w are marked and not adjacent. Let $(G - w)'$ be the graph obtained from $G - w$ by changing the weights of v to $\alpha'(v) = \alpha(v)\alpha(w) + \beta(v)\beta(w)$ and $\beta'(v) = \alpha(v)\beta(w) + \beta(v)\alpha(w)$. Then $[G] = [(G - w)']$.*

(b) *Suppose v and w are marked and adjacent. Let $(G - w)'$ be the graph obtained from $G - w$ by changing the weights of v to $\alpha'(v) = \alpha(v)\alpha(w)$ and $\beta'(v) = \alpha(v)\beta(w) + \beta(v)\alpha(w) + \beta(v)\beta(w)d$. Then $[G] = [(G - w)']$.*

(c) *Suppose v and w are unmarked and adjacent. Let $(G - w)'$ be the graph obtained from $G - w$ by marking v and changing the weights of v to $\alpha'(v) = \alpha(v)\alpha(w)$ and $\beta'(v) = \alpha(v)\beta(w) + \beta(v)\alpha(w) + \beta(v)\beta(w)d$. Then $[G] = [(G - w)']$.*

(d) *Suppose v and w are adjacent, with v unmarked and w marked. Let $(G - w)'$ be the graph obtained from $G - w$ by changing the weights of v to $\alpha'(v) = \alpha(v)\alpha(w)$ and $\beta'(v) = \alpha(v)\beta(w) + \beta(v)\alpha(w) + \beta(v)\beta(w)d$. Then $[G] = [(G - w)']$.*

Proof. Observe that v and w give rise to nearly identical rows and columns of $\mathcal{A}(G)$; they differ only in their common entries, and only if v and w are neighbors. The four parts of the lemma are all justified by applying to $\mathcal{A}(G)_T$ the following nullity calculations over $GF(2)$.

$$\nu \begin{pmatrix} 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & \mathbf{0} & M_{21} & M_{22} \end{pmatrix} - 1 = \nu \begin{pmatrix} 1 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & \mathbf{0} & M_{21} & M_{22} \end{pmatrix} = \nu \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix}$$

$$\nu \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & M_{11} & M_{12} \\ 0 & 0 & M_{21} & M_{22} \end{pmatrix} = \nu \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & M_{11} & M_{12} \\ 0 & 0 & M_{21} & M_{22} \end{pmatrix} = \nu \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\nu \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & M_{11} & M_{12} \\ 0 & 0 & M_{21} & M_{22} \end{pmatrix} - 1 = \nu \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & M_{11} & M_{12} \\ 0 & 0 & M_{21} & M_{22} \end{pmatrix} = \nu \begin{pmatrix} 1 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{pmatrix}$$

For instance, consider part (c). As neither v nor w is marked, the definition of $\mathcal{A}(G)_T$ will not ever involve deleting the row and column corresponding to either. The second nullity calculation shows that if $T \subseteq V(G)$ has $v, w \notin T$ then the term in Definition 1's formula for $[G]$ corresponding to T is the same as the term in Definition 1's formula for $[(G - w)']$ corresponding to T . The third nullity calculation shows that sum of the terms corresponding to $T \cup \{v\}$, $T \cup \{w\}$ and $T \cup \{v, w\}$ in Definition 1's formula for $[G]$ coincides with the term in Definition 1's formula for $[(G - w)']$ corresponding to $T \cup \{v\}$. ■

Observe that the reduced graphs $(G - w)'$ in parts (b) and (c) of Proposition 12 are the same; this is not surprising, as the original graphs differ only by a marked pivot. Also, parts (b)-(d) inductively imply Theorem 3 of the Introduction.

Two cases are missing from Proposition 12. These cases are not fully analogous to series-parallel reductions of edges, as they do not involve the consolidation of two vertices into one vertex of one graph.

Proposition 13 *Let $v, w \in V(G)$ be nonadjacent, unlooped twins.*

(a) *Suppose neither v nor w is marked. Let $(G - w)'$ be the graph obtained from $G - w$ by changing the weights of v to $\alpha'(v) = \alpha(v)\alpha(w)d + \alpha(v)\beta(w) + \beta(v)\alpha(w)$ and $\beta'(v) = 0$. Then $[G] = [(G - w)'] + \beta(v)\beta(w)[G - v - w]$.*

(b) *Suppose v is unmarked and w is marked. Let $(G - w)'$ be the graph obtained from $G - w$ by changing the weights of v to $\alpha'(v) = \alpha(v)\alpha(w) + \alpha(v)\beta(w)$ and $\beta'(v) = \beta(v)\alpha(w)$. Then $[G] = [(G - w)'] + \beta(v)\beta(w)[G - v - w]$.*

Proof. The proofs use the same nullity calculations that appear in the proof of Proposition 12. These cases are more complicated because each one involves both the first nullity calculation and the second, and the corresponding matrices do not appear together in $[G - v]$ or $[G - w]$. ■

The inductive version of Proposition 13 allow us to simplify a graph G containing k nonadjacent twins into one or two graphs with $|V(G)| - k + 1$ or $|V(G)| - k$ vertices. A preliminary observation will be useful.

Lemma 14 *Suppose two graphs G_1 and G_2 are identical except for the weights of a single vertex a , and let G be the graph that is identical to both G_1 and*

G_2 except for $\alpha(a) = r_1\alpha_1(a) + r_2\alpha_2(a)$ and $\beta(a) = r_1\beta_1(a) + r_2\beta_2(a)$. Then $[G] = r_1[G_1] + r_2[G_2]$.

Proof. This follows directly from Definition 1. ■

Corollary 15 Let v_1, \dots, v_k be nonadjacent, unlooped twins in G . Then there are weights $\alpha'(v_1), \beta'(v_1)$ and a coefficient γ so that $[G] = [(G - v_2 - \dots - v_k)'] + \gamma \cdot [G - v_1 - \dots - v_k]$.

Proof. Suppose first that v_1, \dots, v_k are all unmarked and $k \geq 3$. Part (a) of Proposition 13 gives us $[G] = [(G - v_k)'] + \beta(v_{k-1})\beta(v_k)[G - v_{k-1} - v_k]$. As $\beta'(v_{k-1}) = 0$, applying part (a) of Proposition 13 to $(G - v_k)'$ gives us $[(G - v_k)'] = [((G - v_k)' - v_{k-1})']$, where the weights of v_{k-2} in $((G - v_k)' - v_{k-1})'$ have been changed to $\alpha''(v_{k-2}) = \alpha'(v_{k-1})\alpha(v_{k-2})d + \beta(v_{k-2})\alpha'(v_{k-1})$ and $\beta''(v_{k-2}) = 0$. Lemma 14 tells us that $[G] = [((G - v_k)' - v_{k-1})'] + \beta(v_{k-1})\beta(v_k)[G - v_{k-1} - v_k] = [((G - v_k) - v_{k-1})''']$, where the only difference between $(G - v_k) - v_{k-1}'''$ and $G - v_k - v_{k-1}$ is that the weights of v_{k-2} are different. Repeating this process $\lfloor \frac{k-1}{2} \rfloor$ times will ultimately reduce G to a version of $G - v_2 - \dots - v_k$ or $G - v_3 - \dots - v_k$ which has been modified only by changing the weights of v_1 or v_2 . In the former case $\gamma = 0$, and in the latter case one more application of part (a) of Proposition 13 is needed.

Suppose now that $k \geq 3$ and v_k is the only marked vertex among v_1, \dots, v_k . Part (b) of Proposition 13 gives us $[G] = [(G - v_k)'] + \beta(v_{k-1})\beta(v_k)[G - v_{k-1} - v_k]$. Applying part (a) of Proposition 13 then tells us that $[G]$ is

$$\begin{aligned} & \beta'(v_{k-1})\beta(v_{k-2})[(G - v_k)' - v_{k-1} - v_{k-2}] + \\ & [((G - v_k)' - v_{k-1})'] + \beta(v_{k-1})\beta(v_k)[G - v_{k-1} - v_k]. \end{aligned}$$

Lemma 14 tells us that the sum of the second and third terms is the bracket polynomial of a graph that differs from $G - v_{k-1} - v_k$ only in the weights of v_{k-2} . If $k > 3$ then the paragraph above applies to this new graph and also to $(G - v_k)' - v_{k-1} - v_{k-2}$, as neither has any marked vertex among the remaining v_i . The resulting expressions may be combined by using Lemma 14 to get appropriate weights for v_1 and simply adding together the coefficients multiplying $[G - v_1 - \dots - v_k]$.

If there are two or more marked vertices among v_1, \dots, v_k , then we may apply part (a) of Proposition 12 repeatedly to bring the number of marked vertices down to one. ■

The first paragraph of the proof yields a partial “dual” of Theorem 3.

Corollary 16 Suppose $k \geq 3$ is odd and v_1, \dots, v_k are unmarked, nonadjacent, unlooped twins in G . Then $[G] = [(G - v_2 - \dots - v_k)']$ where

$$\beta'(v_1) = \prod_{i=1}^k \beta(v_i) \quad \text{and} \quad \alpha'(v_1) = d^{-1} \left(-\beta'(v_1) + \prod_{i=1}^k (\alpha(v_i)d + \beta(v_i)) \right).$$

4 A recursion for the weighted bracket

A recursion for the bracket polynomial of marked graphs was given in [37]. Modifying this recursion to describe the vertex-weighted version of the bracket is not difficult. Recall that the *open neighborhood* $N(v)$ of a vertex of a graph contains the vertices $w \neq v$ such that $\{v, w\} \in E(G)$.

Definition 17 If $v \in V(G)$ then the local complement G^v has $V(G^v) = V(G)$ and $E(G^v) = \{\{a, b\} \mid a \notin N(v) \text{ and } \{a, b\} \in E(G)\} \cup \{\{a, b\} \mid a, b \in N(v) \text{ and } \{a, b\} \notin E(G)\}$.

That is, G^v is obtained from G by toggling loops and non-loop edges in the subgraph induced by $N(v)$.

Definition 18 If $v, w \in V(G)$ then the pivot G^{vw} is obtained from G by toggling every adjacency between vertices a and b such that $a \in N(v)$, $b \in N(w)$, and either $a \notin N(w)$ or $b \notin N(v)$.

Definition 19 If $v, w \in V(G)$ are neighbors then the marked pivot G_c^{vw} is obtained from G^{vw} by interchanging the neighbors of v and w and toggling the marks on v and w .

None of these three operations affects free loops or the weights of any vertex.

Theorem 20 The weighted bracket polynomial of G satisfies the following.

- (a) Suppose G has ϕ free loops, and let G' be the graph obtained from G by removing the free loops. Then $[G] = d^\phi \cdot [G']$.
- (b) Suppose v is a looped vertex of G , and let G' be the graph obtained from G by removing the loop on v and interchanging $\alpha(v)$ and $\beta(v)$. Then $[G] = [G']$.
- (c) If v and w are neighbors in G then $[G] = [G_c^{vw}]$.
- (d) Suppose v is unlooped and marked, and no neighbor of v is marked. Then

$$[G] = \alpha(v)[G - v] + \beta(v)[G^v - v],$$

where $G - v$ is obtained from G by removing v and every edge incident on v .

- (e) Let v and w be adjacent, unlooped, unmarked vertices. If no neighbor of v is marked then

$$[G] = \alpha(v)\alpha(w)[G^{vw} - v - w] + \alpha(v)\beta(w)[(G^{vw})^v - v - w] + \beta(v)[G^v - v].$$

- (f) If v is an isolated, unlooped, unmarked vertex of G then

$$[G] = (\alpha(v)d + \beta(v)) \cdot [G - v].$$

- (g) The empty graph \emptyset has $[\emptyset] = 1$.

Proof. Parts (a), (b), (f) and (g) follow immediately from Definition 1. Part (c) follows from the fact that for every $T \subseteq V(G)$, the $GF(2)$ -nullities of $\mathcal{A}(G)_T$ and $\mathcal{A}(G_c^{vw})_T$ are the same; see Section 5 of [37]. Parts (d) and (e) are proven just as in the unweighted case; see Section 6 of [37]. ■

Theorem 20 provides a recursive algorithm for the weighted bracket polynomial: first use (a) to eliminate free loops; then use (b) to eliminate loops; then use (c) to eliminate adjacencies between marked vertices; then use (d) to eliminate marked vertices; then use (e) to eliminate the remaining adjacencies; and finally use (f) and (g) to calculate the bracket polynomials of the remaining edgeless, unmarked graphs. Different individual implementations of the algorithm will involve applying parts (c)-(e) at different locations in G , and just as in the unweighted case there is no canonical way to find the most efficient implementation.

Another property of the unweighted bracket that extends directly to the weighted version is this: if G is the union of disjoint subgraphs G_1 and G_2 then $[G] = [G_1] \cdot [G_2]$.

5 Proof of Theorem 6

In outline, our proof of Theorem 6 is similar to the corresponding proof for vertex-weighted interlace polynomials [36]. This similarity is not surprising as weighted interlace polynomials can also be calculated recursively using pivots and local complements. A more complicated argument is required here, however, because interlace polynomials have more convenient reductions for twin and pendant vertices, and do not involve marked vertices.

Suppose that $G = F * H$, and a has no marked neighbor in H . Consider the following calculation of $[G]$.

First, unloop each looped vertex $v \in V(F)$, and reverse $\alpha(v)$ and $\beta(v)$. This does not affect any edges other than loops in $F - a$, so when we are done we have a composition $F' * H$ with no loops in F' .

Second, eliminate adjacencies between marked vertices of $F' - a$ using marked pivots. As no two vertices of H have different nonempty sets of neighbors in F' , these marked pivots will not affect the internal structure of H . However there may be extensive changes within F' . The graph $F'' * H$ resulting from this step is not unique; but any two differ only by marked pivots, and have the same bracket polynomial as G .

As no neighbor of a in H is marked, no marked vertex of F'' has a marked neighbor in $F'' * H$. Consequently part (d) of Theorem 20 may be used to eliminate every marked vertex in F'' . The result is a sum of terms, each of which is the product of an initial factor and some bracket polynomial $[F''' * H]$ or $[F''' * H^a]$.

Fourth, eliminate all remaining adjacencies in each $F''' - a$ using part (e) of Theorem 20. As each resulting term is obtained by applying local complement or pivot operations, each term is the product of an initial factor and a

bracket polynomial $[F'''' * H]$ or $[F'''' * H^a]$, with F'''' unmarked and edgeless. Consequently every vertex not adjacent to a in any F'''' is now unmarked and isolated; part (f) of Theorem 20 tells us that each term is unchanged if we remove all such vertices from that term's F''' and multiply the initial factor appropriately. If a term $[F'''' * H]$ involves a graph F'''' in which no vertex neighbors a , we may replace that term with a bracket polynomial $[P_{1m} * H]$, where $V(P_{1m}) = \{a, a_m\}$, a and a_m are adjacent, and a_m is a new marked vertex with $\beta(a_m) = 0$ and $\alpha(a_m) = 1$. Similarly, any term $[F'''' * H^a]$ in which F'''' contains no neighbor of a may be replaced with a bracket polynomial $[\tilde{P}_{1m} * H^a]$, where $V(\tilde{P}_{1m}) = \{a, \tilde{a}_m\}$, a and \tilde{a}_m are adjacent, and \tilde{a}_m is a new marked vertex with $\beta(\tilde{a}_m) = 0$ and $\alpha(\tilde{a}_m) = 1$. After these manipulations $[G]$ is expressed as a sum of terms each of which is the product of an initial factor and a weighted bracket polynomial $[F'''' * H]$ or $[F'''' * H^a]$ in which $F'''' - a$ is a nonempty collection of isolated unlooped neighbors of a .

Fifth, apply Proposition 13 and part (a) of Proposition 12 repeatedly to each term in which $F'''' - a$ contains more than one vertex, ultimately obtaining an expression of $[G]$ as a sum of terms each of which is the product of an initial factor with a weighted bracket $[F! * H]$ or $[F! * H^a]$ where $F! - a$ consists of a single unlooped neighbor of a , denoted a_u, a_m, \tilde{a}_u or \tilde{a}_m according to whether that vertex is unmarked or marked (subindex u or m) and whether that term involves H or H^a (the latter indicated by tilde). The value of an individual term is not changed if we multiply both the α and β weights of that term's vertex a_x or \tilde{a}_x by the initial factor, and then replace the initial factor with 1; consequently we may as well presume that the initial factors are all 1. Using Lemma 14, we see that if we sum the α and β weights of a_u, \tilde{a}_u, a_m and \tilde{a}_m in all the different terms involving each, then $[G]$ is equal to the total of four individual weighted bracket polynomials:

$$[G] = [P_1 * H] + [\tilde{P}_1 * H^a] + [P_{1m} * H] + [\tilde{P}_{1m} * H^a]. \quad (1)$$

Here each of the graphs $P_1, \tilde{P}_1, P_{1m}, \tilde{P}_{1m}$ consists of a and a single neighbor of a , denoted $a_u, \tilde{a}_u, a_m, \tilde{a}_m$ respectively as before.

We now seem to have the job of determining eight unknowns, namely the α and β weights of the four vertices $a_u, \tilde{a}_u, a_m, \tilde{a}_m$. It turns out though that five of these unknowns are not necessary.

For instance, $\beta(\tilde{a}_m)$ appears in terms of $[\tilde{P}_{1m} * H^a]$ that involve the $GF(2)$ -nullities of matrices of the form

$$\begin{pmatrix} 1 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & M_{11} & M_{12} \\ \mathbf{1} & M_{21} & M_{22} \end{pmatrix}, \text{ where } \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \mathcal{A}(H^a)_T.$$

As H contains no marked neighbor of a , for each such matrix

$$\nu \begin{pmatrix} 1 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & M_{11} & M_{12} \\ \mathbf{1} & M_{21} & M_{22} \end{pmatrix} = \nu \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \nu(\mathcal{A}(H)_T),$$

where the overbar indicates that every entry of \bar{M}_{22} is different from the corresponding entry of M_{22} . (N.b. If we did not know that the neighbors of a in H are unmarked, the last equality would be suspect as the definition of $\mathcal{A}(H)_T$ would involve removing some rows of \bar{M}_{22} and retaining some rows removed in the definition of $\mathcal{A}(H^a)_T$.) Consequently the contribution to the sum of (1) made by the terms in which $\beta(\tilde{a}_m)$ appears may be provided equally well by terms of $[P_{1m} * H]$ in which $\alpha(a_m)$ appears. That is, the sum is unchanged if we replace $\alpha(a_m)$ by $\alpha(a_m) + \beta(\tilde{a}_m)$ and replace $\beta(\tilde{a}_m)$ by 0.

The terms of $[\tilde{P}_1 * H^a]$ in which $\beta(\tilde{a}_u)$ appears involve the $GF(2)$ -nullities of precisely the same matrices just discussed. Consequently the contributions to the sum of (1) made by the terms of $[\tilde{P}_1 * H^a]$ in which $\beta(\tilde{a}_u)$ appears may also be provided by the terms of $[P_{1m} * H]$ in which $\alpha(a_m)$ appears; that is, we may replace $\alpha(a_m)$ by $\beta(\tilde{a}_u) + \alpha(a_m)$ and replace $\beta(\tilde{a}_u)$ by 0.

The equality

$$\nu \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \nu \begin{pmatrix} 1 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & M_{11} & M_{12} \\ \mathbf{1} & M_{21} & \bar{M}_{22} \end{pmatrix}$$

tells us that the contributions of the terms in which $\alpha(\tilde{a}_m)$ appears can be duplicated by the terms in which $\beta(a_u)$ appears, i.e., the sum of (1) is unchanged if we replace $\beta(a_u)$ by $\beta(a_u) + \alpha(\tilde{a}_m)$ and replace $\alpha(\tilde{a}_m)$ by 0. With both weights of \tilde{a}_m now 0, $[\tilde{P}_{1m} * H^a] = 0$ makes no contribution to (1).

In the same way, the equality

$$\nu \begin{pmatrix} 0 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & M_{11} & M_{12} \\ \mathbf{1} & M_{21} & \bar{M}_{22} \end{pmatrix} = \nu \begin{pmatrix} 0 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & M_{11} & M_{12} \\ \mathbf{1} & M_{21} & M_{22} \end{pmatrix}$$

tell us that \tilde{a}_u is not needed: the contributions of the terms of (1) involving $\alpha(\tilde{a}_u)$ may be provided by the terms involving $\alpha(a_u)$.

The terms of (1) in which $\beta(a_m)$ appears involve the same nullities

$$\nu \begin{pmatrix} 1 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & M_{11} & M_{12} \\ \mathbf{1} & M_{21} & M_{22} \end{pmatrix}$$

that appear in the terms in which $\beta(a_u)$ appears, so the sum of (1) is unchanged if we replace $\beta(a_u)$ with $\beta(a_u) + \beta(a_m)$ and replace $\beta(a_m)$ with 0.

$P_1 * H$ and $P_{1m} * H$ are isomorphic to H' and H'_m respectively, so the statement of Theorem 6 follows.

6 Subset formulas for the weights of Theorem 6

In this section we use linear algebra over $GF(2)$ to derive formulas for the weights $\alpha(a_u)$, $\beta(a_u)$ and $\alpha(a_m)$ of Theorem 6 from Definition 1.

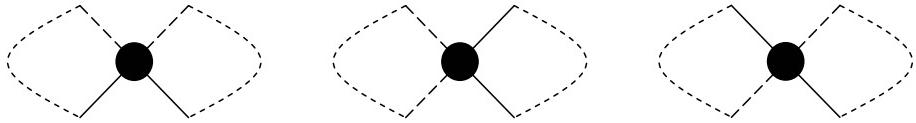


Figure 10: The circuit partition pictured in the middle contains one more circuit.

Suppose $T \subseteq V(F - a)$, and let i_1, \dots, i_k be the indices of the rows and columns not removed from $\mathcal{A}(F - a)$ in obtaining $\mathcal{A}(F - a)_T$. That is, $V(F - a) - \{v_{i_1}, \dots, v_{i_k}\} = \{\text{marked } v \in V(F - a) \mid \text{either } v \in T \text{ is looped or } v \notin T \text{ is not looped}\}$. Let $\rho = (\rho_{i_1} \dots \rho_{i_k})$ be the row vector with $\rho_{i_j} = 1$ if v_{i_j} is a neighbor of a , and let κ be the column vector obtained by transposing ρ . According to Lemma 2 of [2], the three nullities

$$\nu \begin{pmatrix} \mathcal{A}(F)_T & \kappa \\ \rho & 0 \end{pmatrix}, \nu \begin{pmatrix} \mathcal{A}(F)_T & \kappa \\ \rho & 1 \end{pmatrix}, \nu(\mathcal{A}(F)_T)$$

are of the form $\nu + 1, \nu, \nu$ in some order. We say T is of *type 1, 2 or 3* according to whether the nullity $\nu + 1$ appears first, second or third.

If D is a link diagram, then the three displayed matrices correspond to three circuit partitions in U . At every vertex other than a , the three partitions have the same transition. In this situation Lemma 2 of [2] asserts that two of the partitions contain the same number of circuits, and the third contains one more circuit. An example of type 2 is illustrated schematically in Figure 10.

For each subset $T \subseteq V(F - a)$ let $\text{contr}(T)$ denote the contribution of T to Definition 1's formula for $[F - a]$,

$$(\prod_{v \notin T} \alpha(v))(\prod_{t \in T} \beta(t))d^{\nu(\mathcal{A}(G)_T)}.$$

If T is of type 1 then the sum of the contributions of T and $T \cup \{v_{10}\}$ to $[F^{10}]$ is $d \cdot \text{contr}(T)$, while the sum of the contributions of T and $T \cup \{v_{01}\}$ to $[F^{01}]$ is $\text{contr}(T)$. If T is of type 2 then the sum of the contributions of T and $T \cup \{v_{10}\}$ to $[F^{10}]$ is $\text{contr}(T)$, and the sum of the contributions of T and $T \cup \{v_{01}\}$ to $[F^{01}]$ is $d \cdot \text{contr}(T)$. If T is of type 3 then the sum of the contributions of T and $T \cup \{v_{10}\}$ to $[F^{10}]$ is $(1/d) \cdot \text{contr}(T)$, and the sum of the contributions of T and $T \cup \{v_{01}\}$ to $[F^{01}]$ is also $(1/d) \cdot \text{contr}(T)$. As noted in the Introduction, Theorem 6 implies these equations.

$$\begin{aligned} (2 - d - d^2)\alpha(a) &= -(d + 1)[F - a] + [F^{10}] + [F^{01}] \\ (2 - d - d^2)\beta(a) &= [F - a] + [F^{10}] - (d + 1)[F^{01}] \\ (2 - d - d^2)\alpha(a_m) &= [F - a] - (d + 1)[F^{10}] + [F^{01}] \end{aligned}$$

It follows that if we denote the total of the contributions of the sets of type i to $[F - a]$ by $contr_i$, then these equalities hold.

$$\begin{aligned}(2 - d - d^2)\alpha(a) &= -(d + 1) + (2/d) \cdot contr_3 \\ (2 - d - d^2)\beta(a) &= (2 - d(d + 1)) \cdot contr_2 \\ (2 - d - d^2)\alpha(a_m) &= (2 - d(d + 1)) \cdot contr_1\end{aligned}$$

Corollary 21 *The weights of Theorem 6 are $\alpha(a_m) = contr_1$, $\beta(a) = contr_2$ and $\alpha(a) = (contr_3)/d$, where*

$$contr_i = \sum_{\substack{T \subseteq V(F-a) \\ \text{of type } i}} \left(\prod_{v \notin T} \alpha(v) \right) \left(\prod_{t \in T} \beta(t) \right) d^{\nu(\mathcal{A}(G)_T)}.$$

7 Some examples

The simplest example of Theorem 6 involves the two-vertex graph F in which a and v are unlooped, unmarked neighbors. Let $\alpha = \alpha(v)$ and $\beta = \beta(v)$. For this graph Corollary 7 gives the following.

$$\begin{aligned}(2 - d - d^2)\alpha(a) &= -(d + 1)(\alpha d + \beta) + \alpha + \beta + \alpha + \beta d = (2 - (d + 1)d)\alpha \\ (2 - d - d^2)\beta(a) &= \alpha d + \beta + \alpha + \beta - (d + 1)(\alpha + \beta d) = (2 - (d + 1)d)\beta \\ (2 - d - d^2)\alpha(a_m) &= \alpha d + \beta - (d + 1)(\alpha + \beta) + \alpha + \beta d = 0\end{aligned}$$

The assertion of Theorem 1 is then trivial, as $F * H$ and H' are identical graphs.

A slightly more complicated example involves the same graph F , but with v marked; Corollary 7 gives the following.

$$\begin{aligned}(2 - d - d^2)\alpha(a) &= -(d + 1)(\alpha + \beta) + \alpha d + \beta + \alpha + \beta d = 0 \\ (2 - d - d^2)\beta(a) &= \alpha + \beta + \alpha d + \beta - (d + 1)(\alpha + \beta d) = (2 - (d + 1)d)\beta \\ (2 - d - d^2)\alpha(a_m) &= \alpha + \beta - (d + 1)(\alpha d + \beta) + \alpha + \beta d = (2 - (d + 1)d)\alpha\end{aligned}$$

This seems more interesting than the result of the first example, but it is just as trivial. In the last step of the proof of Theorem 6 we see that setting $\beta(a_m) = 0$ as in the statement of the theorem is arbitrary; any choice of $\beta(a_m)$ will satisfy the theorem, so long as the sum $\beta(a) + \beta(a_m)$ is correct. With $\beta(a) = 0$ instead, the assertion of Theorem 6 simply acknowledges that in this example, $F * H$ and H'_m are identical.

Let D be the link diagram pictured in the middle of Figure 11. Then $\mathcal{L}(D, C) = F * H$, where $F - a, H - a$ are the pictured two-vertex graphs and a is adjacent to all four of these vertices. Consider both $F - a$ and $H - a$ to have the standard weight functions $\alpha \equiv A$ and $\beta \equiv B$. Corollary 7 tells us that $\alpha(a) = A^2$, $\beta(a) = 2AB + B^2d$ and $\alpha(a_m) = 0$. Theorem 6 then asserts that

$$\begin{aligned}[F * H] &= [H'] = \\ A^2 \cdot (A^2d + 2AB + B^2d) + (2AB + B^2d) \cdot (A^2d^2 + 2ABd + B^2).\end{aligned}$$

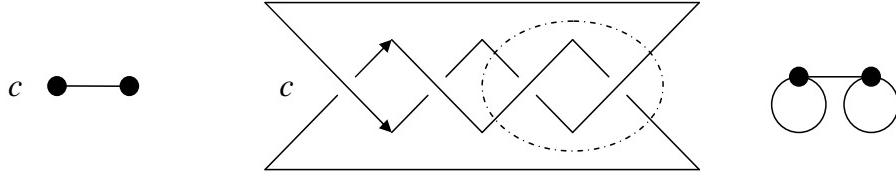


Figure 11: A two-component unlink. $F - a$ is on the left and $H - a$ is on the right.

(The same conclusion follows from Theorem 3.) The writhe of D is 0, so this bracket polynomial determines the Jones polynomial through the following two-stage evaluation. First $d \mapsto -A^2 - B^2$ and $B \mapsto A^{-1}$ yield

$$f_D(A) = A^2 \cdot (-A^4 - A^{-4}) + (1 - A^{-4}) \cdot (A^6) = -A^{-2} - A^2,$$

and then $V = f_D(t^{-1/4}) = -t^{1/2} - t^{-1/2}$, as we would expect for the two-component unlink.

Suppose we modify the diagram in Figure 11 by reversing the crossing on the right, effectively removing one loop from $H - a$. Then

$$\begin{aligned} [F * H] &= [H'] = \\ A^2 \cdot (A^2 + 2ABd + B^2) &+ (2AB + B^2d) \cdot (A^2d + AB + ABd^2 + B^2d). \end{aligned}$$

The writhe of the diagram is now 2, so the Jones polynomial is obtained by first calculating $f_D(A)$:

$$\begin{aligned} A^{-6} \cdot (A^4 + 2(-A^4 - 1) + 1 + (1 - A^{-4}) \cdot (-A^4 + A^4 + 2 + A^{-4} - 1 - A^{-4})) \\ = A^{-6} \cdot (-A^4 - A^{-4}), \end{aligned}$$

and then evaluating $f_D(t^{-1/4}) = t^{3/2} \cdot (-t - t^{-1})$, the correct value for the positive Hopf link.

Suppose instead we modify the graph $F - a$ pictured in Figure 11 by removing the mark. Corollary 7 tells us that $\alpha(a) = 0$, $\beta(a) = 2AB + B^2d$ and $\alpha(a_m) = A^2$. According to Theorem 6,

$$\begin{aligned} [F * H] &= [H'] + [H'_m] = \\ (2AB + B^2d) \cdot (A^2d^2 + 2ABd + B^2) &+ A^2 \cdot (A^2d + 2AB + B^2). \end{aligned}$$

As the writhe is 0, this yields

$$f_D(A) = (1 - A^{-4}) \cdot (A^6) + A^2 \cdot (1 - A^4 + A^{-2}) = 1.$$

This seems incorrect at first glance, because Figure 11 displays a two-component link. Note however that although the graph obtained by removing the mark from

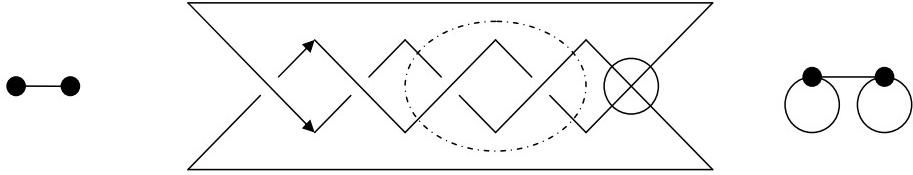


Figure 12: An unknot. $F - a$ is on the left and $H - a$ is on the right.

$F - a$ is certainly a legitimate marked graph, we cannot legitimately remove the mark from the link diagram in Figure 11, because the marks on a connected link diagram must identify an Euler circuit. Figure 12 exhibits an unmarked link diagram with the appropriate graphs $F - a$ and $H - a$; it is an unknot rather than a two-component unlink, so $V = 1$ is indeed its Jones polynomial.

Suppose we now modify the diagram in Figure 11 by reversing the orientation of one component, as in Figure 13. The only vertex of F adjacent to a is the marked vertex, while both vertices of $H - a$ are neighbors of a . Corollary 7 tells us that $\alpha(a) = A^2$, $\beta(a) = 0$ and $\alpha(a_m) = 2AB + B^2d$. Then Theorem 6 asserts that

$$[F * H] = [H'] + [H'_m] = \\ A^2 \cdot (A^2d + 2AB + B^2d) + (2AB + B^2d) \cdot (A^2d^2 + 2ABd + B^2),$$

just as in the discussion of Figure 11 above. This is not surprising, as the Kauffman bracket of a link diagram is independent of the orientations of the link components.

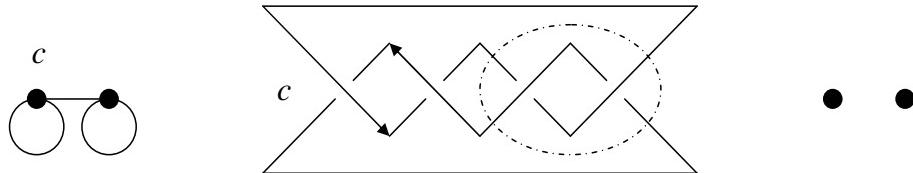


Figure 13: A two-component unlink. The marked vertex on the left is the only neighbor of a in F ; a is also adjacent to the two vertices of $H - a$ on the right.

In our last example we consider the unmarked version of the example pictured in Figure 5. Direct calculations yield the following.

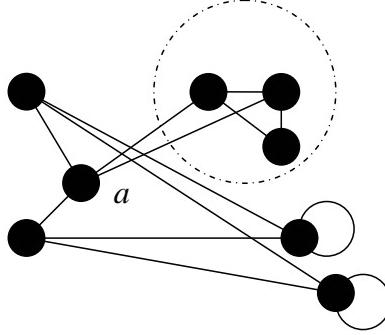


Figure 14: The unmarked graphs F and H from Figure 5, with $H - a$ inside the dashed circle and F outside the dashed circle.

$$\begin{aligned}
 [F - a] &= A^4d^2 + 4A^3Bd + 5A^2B^2 + A^2B^2d^2 + 4AB^3d + B^4d^2 \\
 [F^{01}] &= A^4d + 2A^3B + 2A^3Bd + 5A^2B^2 + A^2B^2d^2 + 4AB^3d + B^4d^2 \\
 [F^{10}] &= A^4d + 2A^3B + 2A^3Bd^2 + 5A^2B^2d + A^2B^2d^3 + 4AB^3d^2 + B^4d^3 \\
 [H - a] &= A^3d + 3A^2B + 3AB^2d + B^3d^2 \\
 [H^{01}] &= A^3d + 3A^2B + AB^2d + 2AB^2 + B^3d \\
 [H^{10}] &= A^3d^2 + 3A^2Bd + AB^2d^2 + 2AB^2 + B^3d
 \end{aligned}$$

Corollary 7 gives us these weights.

$$\begin{aligned}
 \alpha(a(F)) &= A^4d + 2A^3B, \beta(a(F)) = 0, \\
 \alpha(a_m(F)) &= 2A^3Bd + A^2B^2(5 + d^2) + 4AB^3d + B^4d^2, \\
 \alpha(a(H)) &= 2AB^2 + B^3d, \beta(a(H)) = 0, \alpha(a_m(H)) = A^3d + 3A^2B + AB^2d
 \end{aligned}$$

Corollary 8 tells us that

$$\begin{aligned}
 [F * H] &= (A^4d + 2A^3B) \cdot (2AB^2 + B^3d) + 0 \\
 &\quad + ((A^4d + 2A^3B)d + 0) \cdot (A^3d + 3A^2B + AB^2d) \\
 &\quad + (2A^3Bd + A^2B^2(5 + d^2) + 4AB^3d + B^4d^2) \cdot ((2AB^2 + B^3d)d + 0) \\
 &\quad + (2A^3Bd + A^2B^2(5 + d^2) + 4AB^3d + B^4d^2) \cdot (A^3d + 3A^2B + AB^2d).
 \end{aligned}$$

The writhe of D is 3, so we calculate $f_D(A)$ by multiplying by $-A^{-9}$ and

evaluating $d \mapsto -A^2 - B^2$ and $B \mapsto A^{-1}$:

$$\begin{aligned}
f_D(A) &= -A^{-9} \cdot (-A^6 + A^2) \cdot (A^{-1} - A^{-5}) \\
&\quad - A^{-9} \cdot (-A^6 + A^2)(-A^2 - A^{-2}) \cdot (-A^5 + A - A^{-3}) \\
&\quad - A^{-9} \cdot (-A^4 + 2 - A^{-4} + A^{-8}) \cdot (-A^{-5} + A^{-1})(-A^2 - A^{-2}) \\
&\quad - A^{-9} \cdot (-A^4 + 2 - A^{-4} + A^{-8}) \cdot (-A^5 + A - A^{-3}) \\
&= -A^{-9} \cdot (-A^{13} + 2A^9 - 3A^5 + 3A - 4A^{-3} + 3A^{-7} - 2A^{-11} + A^{-15})
\end{aligned}$$

We conclude that the Jones polynomial of the knot of Figure 5 is $f_D(t^{-1/4}) = -t^6 + 2t^5 - 3t^4 + 4t^3 - 3t^2 + 3t - 2 + t^{-1}$. This identifies the knot as the mirror image of 7_6 .

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